

# ANALYSIS OF THREE FREQUENCY UNDULATOR INTENSITY AND GAIN DUE TO OFF AXIS CONTRIBUTION IN FREE ELECTRON LASER

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## ABSTRACT

*In this paper we study the three frequency undulator radiation and free electron laser gain for higher harmonics with the inclusion of off axis contribution. When electron enters the field off axis it causes additional oscillations due to which intensity and gain reduction take place. To enhance the intensity and gain, we introduce a new scheme -the three frequency undulator scheme.*

**Keywords:** Free electron laser, intensity, undulator

**PACS:** 41.60.Cr

## 1. Introduction

In a free electron laser (FEL) [1-15], a relativistic beam of electrons passes through a periodic transverse magnetic field, to produce coherent radiation. The main advantage of the FEL is that it is tunable. In standard FEL, electron passes through on-axis undulator field.

When electron enters on-axis, the interaction region of undulators of a free electron laser device, it sees an on-axis field and executes small oscillations around the axis. However this on-axis field does not satisfy Maxwell's equation. Hence the off-axis fields are calculated from the Maxwell equations. With the presence of these off-axis fields, the electron executes additional oscillations known as betatron oscillations. These extra oscillations drive away electrons from the resonance and FEL gain drops. Both off-axis and angular injection of the electron induces betatron oscillations. When it enters the field off-axis causes additional field components of the undulator field. And this extra additional oscillation causes the degradation in the intensity and gain free electron laser.

The two-frequency & two-harmonic undulator [16-21], Optical Klystron Undulator [23-27] are some examples which have attracted wide interests in this context. In this paper we analyze the case of a three frequency [28] undulator scheme to increase the intensity and gain of a free electron laser in the presence of betatron oscillations.

## 2. Undulator Radiation

We assume the electron moves on axis in a three frequency undulator scheme whose on-axis field is given by,

$$\vec{B} = \left[ 0, B_0 a_1 \left\{ \sin(k_u z) + \frac{a_2}{a_1} \sin(2k_u z) + \frac{a_3}{a_1} \sin(3k_u z) \right\}, 0 \right]$$

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Where  $k_u = 2\pi/\lambda_u$ ,  $\lambda_u$  is the undulator wavelength,  $B_0$  is peak field strength. The trajectory of the electron is determined through the Lorentz equation. This gives

$$x(t) = - \left[ \frac{Kc}{\gamma\omega_u} \sin(\omega_u t) + \frac{Kc\delta_1}{2\gamma\omega_u} \sin(2\omega_u t) + \frac{Kc\delta_2}{3\gamma\omega_u} \sin(3\omega_u t) \right] \quad (2)$$

where,  $K = \frac{eB_0 a_1}{m_0 c \omega_u}$ ,  $\delta_1 = (a_2/2a_1)$ ,  $\delta_2 = (a_3/3a_1)$   $\omega_u = k_u c$ , where  $K$  defines the undulator parameter. The z-motion is,

$$\begin{aligned} z(t) = & \beta^* ct - \frac{cK^2}{8\gamma^2\omega_u} \sin(2\omega_u t) - \frac{cK^2\delta_1^2}{16\gamma^2\omega_u} \sin(4\omega_u t) - \frac{cK^2\delta_2^2}{24\gamma^2\omega_u} \sin(6\omega_u t) \\ & - \frac{cK^2\delta_2}{4\gamma^2\omega_u} \sin(2\omega_u t) - \frac{cK^2\delta_1}{6\gamma^2\omega_u} \sin(3\omega_u t) - \frac{cK^2\delta_2}{8\gamma^2\omega_u} \sin(4\omega_u t) - \frac{cK^2\delta_1\delta_2}{10\gamma^2\omega_u} \sin(5\omega_u t) \\ & - \frac{cK^2\delta_1(1+\delta_2)}{2\gamma^2\omega_u} \sin(\omega_u t) \end{aligned} \quad (3)$$

where,

$$\beta^* = 1 - \frac{1}{2\gamma^2} \left[ 1 + \frac{K^2}{2} + \frac{K^2\delta_1^2}{2} + \frac{K^2\delta_2^2}{2} \right]$$

The representation of the field in Eq.(1) is valid if the electron's path remains near to the undulator axis where there are no betatron oscillations. The dependence of the field on the transverse coordinates is found from the Maxwell equations. At the lowest order in transverse coordinates, the field components are [8-9]

$$\begin{aligned} B_x &= \frac{B_0}{2} xy \left[ k_u^2 \delta \{a_1 \sin(k_u z)\} + k_h^2 \sigma \{a_2 \sin(k_h z)\} \right] \\ B_y &= B_0 \left[ 1 + \frac{k_u^2}{4} \{\delta x^2 + (2 - \delta)y^2\} \right] a_1 \sin(k_u z) \\ &\quad + B_0 \left[ 1 + \frac{k_h^2}{4} \{\sigma x^2 + (2 - \sigma)y^2\} \right] a_2 \sin(k_h z) \\ B_z &= B_0 y [a_1 k_u \cos(k_u z) + a_2 k_h \cos(k_h z)] \end{aligned} \quad (4)$$

where  $\delta = 2\alpha^2/k_u^2$  and  $\sigma = 2\alpha^2/k_h^2$ . The equation of motion can now be derived using the field equation in Eq. (4). We assume that the motion can be decomposed as  $x = x_0 + x_1$  where  $x_0$  represents the reference trajectory due to the field and  $x_1$  describe the additional motion around the reference trajectory due to the extra terms in eq(4) depending on the transverse coordinate. Keeping the contributions at the first order only in  $x_1$  and averaging over one undulator period, we get the following differential equation ruling the additional motion,

$$\frac{d^2 x_1}{dt^2} + \Omega_\beta^2 x_1 = 0 \quad (5)$$

where,

$$\Omega_{\beta}^2 = \frac{K_1^2 c^2 k_u^2}{4\gamma^2} [\delta + h^2 \sigma]$$

The solution of Eq.(5) is given by,

$$x_1(t) = x_0 \cos(\Omega_{\beta} t) \quad (6)$$

Where  $x_0$  represent the off-axis position from the undulator axis and  $y_0 = 0$ . The z-motion is

$$\begin{aligned} z(t) = & \beta^{**} ct - \frac{cK^2}{8\gamma^2 \omega_u} \sin(2\omega_u t) - \frac{cK^2 \delta_1^2}{16\gamma^2 \omega_u} \sin(4\omega_u t) - \frac{cK^2 \delta_2^2}{24\gamma^2 \omega_u} \sin(6\omega_u t) \\ & - \frac{cK^2 \delta_2}{4\gamma^2 \omega_u} \sin(2\omega_u t) - \frac{cK^2 \delta_1}{6\gamma^2 \omega_u} \sin(3\omega_u t) - \frac{cK^2 \delta_2}{8\gamma^2 \omega_u} \sin(4\omega_u t) - \frac{cK^2 \delta_1 \delta_2}{10\gamma^2 \omega_u} \sin(5\omega_u t) \\ & - \frac{cK^2 \delta_1(1+\delta_2)}{2\gamma^2 \omega_u} \sin(\omega_u t) + \frac{x_1^2 \Omega_{\beta_x}}{8c^2} \sin(2\Omega_{\beta_x} t) \pm \frac{Kx_1 \Omega_{\beta_x}}{2\gamma c (\omega_u \pm \Omega_{\beta_x})} \cos(\omega_u \pm \Omega_{\beta_x}) t \\ & \pm \frac{Kx_1 \Omega_{\beta_x} \delta_1}{2\gamma c (2\omega_u \pm \Omega_{\beta_x})} \cos(2\omega_u \pm \Omega_{\beta_x}) t \pm \frac{Kx_1 \Omega_{\beta_x} \delta_2}{2\gamma c (3\omega_u \pm \Omega_{\beta_x})} \cos(3\omega_u \pm \Omega_{\beta_x}) t \end{aligned} \quad (7)$$

$$\beta^{**} = 1 - \frac{1}{2\gamma^2} \left[ 1 + \frac{K^2}{2} + \frac{K^2 \delta_1^2}{2} + \frac{K^2 \delta_2^2}{2} + \frac{x_1^2 \Omega_{\beta_x}^2 \gamma^2}{2c^2} \right]$$

The spectral properties of the undulator radiation are easily obtained from the Lienard-Wiechert integral [27],

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2 \omega^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} \{ \hat{n} \times (\hat{n} \times \vec{\beta}) \} \exp \left[ i\omega \left( t - \frac{z}{c} \right) \right] dt \right|^2 \quad (8)$$

The brightness expression reduced to

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2 \omega^2 T^2}{4\pi^2 c} |F_m(z)|^2 \left( \frac{\sin(\nu_0/2)}{\nu_0/2} \right)^2 \quad (9)$$

where,

$$\begin{aligned} F_m(z) = & \left[ \frac{K}{2\gamma} \{ J_{m-1}(0, z_1) + J_{m+1}(0, z_1) \} J_n(0, z_2) J_p(0, z_3) J_q(0, z_4) \right. \\ & + \frac{K\delta_1}{2\gamma} \{ J_{n-1}(0, z_2) + J_{n+1}(0, z_2) \} J_m(0, z_1) J_p(0, z_3) J_q(0, z_4) \\ & + \left. \frac{K\delta_2}{2\gamma} \{ J_{p-1}(0, z_3) + J_{p+1}(0, z_3) \} J_m(0, z_1) J_n(0, z_2) J_q(0, z_4) \right] \end{aligned}$$

$$+ \frac{x_0 \Omega_\beta}{2ic} \left\{ J_{q+1}(0, z_4) - J_{q-1}(0, z_4) \right\} J_m(0, z_1) J_n(0, z_2) J_p(0, z_3) \Bigg] \\ \times J_r(z_5, z_6) J_s(z_7, 0) J_{p1}(z_8, 0) J_{p2}(z_9, 0) J_{p3}(z_{10}, 0) J_{p4}(z_{11}, 0) J_{p5}(z_{12}, 0)$$

$$\nu_0 = \frac{\omega}{2\gamma^2} \left[ 1 + \frac{K^2}{2} + \frac{K^2 \delta_1^2}{2} + \frac{K^2 \delta_2^2}{2} + \frac{x_1^2 \Omega_{\beta_x}^2 \gamma^2}{2c^2} \right] - \eta$$

$$\eta = m\omega_u + n2\omega_u + p3\omega_u t + q\Omega_{\beta_x} r\omega_u + s\frac{3}{2}\omega_u + p_1 2\omega_u + p_2 \frac{5}{2}\omega_u + p_3 (\omega_u \pm \Omega_{\beta_x}) \\ + p_4 (2\omega_u \pm \Omega_{\beta_x}) + p_5 (3\omega_u \pm \Omega_{\beta_x})$$

with

$$z_1 = -\frac{\omega K^2}{8\gamma^2 \omega_u}, z_2 = -\frac{\omega K^2 \delta_1^2}{16\gamma^2 \omega_u}, z_3 = -\frac{\omega K^2 \delta_2^2}{24\gamma^2 \omega_u}, z_4 = \frac{\omega x_1^2 \Omega_{\beta_x}}{8c^2}, z_5 = -\frac{\omega K^2 \delta_1 (1 + \delta_2)}{2\gamma^2 \omega_u}, \\ z_6 = -\frac{\omega K^2 \delta_2}{4\gamma^2 \omega_u}, z_7 = -\frac{\omega K^2 \delta_1}{6\gamma^2 \omega_u}, z_8 = -\frac{\omega K^2 \delta_2}{8\gamma^2 \omega_u}, z_9 = -\frac{\omega K^2 \delta_1 \delta_2}{10\gamma^2 \omega_u}, z_{10} = \pm \frac{K x_1 \Omega_{\beta_x}}{2\gamma c (\omega_u \pm \Omega_{\beta_x})}, \\ z_{11} = \pm \frac{K x_1 \Omega_{\beta_x} \delta_1}{2\gamma c (2\omega_u \pm \Omega_{\beta_x})}, z_{12} = \pm \frac{K x_1 \Omega_{\beta_x} \delta_2}{2\gamma c (3\omega_u \pm \Omega_{\beta_x})}$$

$J_m(0, z_1)$  ... are the generalized Bessel functions of order  $i = m, n, p, q, r, s, p_1, p_2, p_3, p_4, p_5$  respectively. The resonance condition in a free electron laser is provided by  $\nu_0 = 0$ . This provides the central emission frequency as,

$$\omega_i = \frac{2\gamma^2 \eta}{\left[ 1 + \frac{K^2}{2} + \frac{K^2 \delta_1^2}{2} + \frac{K^2 \delta_2^2}{2} + \frac{x_1^2 \Omega_{\beta_x}^2 \gamma^2}{2c^2} \right]}$$

### 3. GAIN

To calculate the gain, let us consider a linear polarized electromagnetic wave with

$$\vec{E} = E_0 \hat{x} \cos(kz - \omega t + \varphi) \quad (10)$$

The change in energy of the electron is given by,

$$\frac{d\gamma}{d\tau} = \frac{e E_o K L_u}{m_0 c^2 \gamma} \cos(\zeta_0 + \varphi) J_q(0, z_4) J_r(z_5, z_6) J_s(z_7, 0) J_{p1}(z_8, 0) J_{p2}(z_9, 0) J_{p3}(z_{10}, 0) J_{p4}(z_{11}, 0) J_{p5}(z_{12}, 0) \times \\ \left[ \{J_{m-1}(0, z_1) + J_{m+1}(0, z_1)\} J_n(0, z_2) J_p(0, z_3) + \delta_1 \{J_{n-1}(0, z_2) + J_{n+1}(0, z_2)\} J_m(0, z_1) J_p(0, z_3) \right] \\ + \delta_2 \{J_{p-1}(0, z_3) + J_{p+1}(0, z_3)\} J_m(0, z_1) J_n(0, z_2) \quad (11)$$

where

$$\zeta_0 = \eta[(k_1 + k_u)z - \omega_1 t]$$

The pendulum equation is,

$$\frac{d^2\zeta_0}{d\tau^2} = |a| \cos(\zeta_0 + \varphi) \quad (12)$$

where the dimensionless field strength is,

$$|a| = \frac{4\pi NeE K_1 L}{\gamma^2 m_0 c^2} \eta J_q(0, z_4) J_r(z_5, z_6) J_s(z_7, 0) J_{p1}(z_8, 0) J_{p2}(z_9, 0) J_{p3}(z_{10}, 0) J_{p4}(z_{11}, 0) J_{p5}(z_{12}, 0) \times \\ \left[ \{J_{m-1}(0, z_1) + J_{m+1}(0, z_1)\} J_n(0, z_2) J_p(0, z_3) + \delta_1 \{J_{n-1}(0, z_2) + J_{n+1}(0, z_2)\} J_m(0, z_1) J_p(0, z_3) \right] \\ + \delta_2 [J_{p-1}(0, z_3) + J_{p+1}(0, z_3)] J_m(0, z_1) J_n(0, z_2) \quad (13)$$

In its simplest form the wave equation is written as,

$$\frac{da}{d\tau} = -j \langle e^{-i\zeta_0} \rangle \quad (14)$$

where  $j$  is the dimensionless current density given as,

$$j = \frac{4\pi^2 Ne^2 K_1^2 L^2 n_e}{\gamma^3 m_0 c^2} \eta J_q(0, z_4) J_r(z_5, z_6) J_s(z_7, 0) J_{p1}(z_8, 0) J_{p2}(z_9, 0) J_{p3}(z_{10}, 0) J_{p4}(z_{11}, 0) J_{p5}(z_{12}, 0) \times \\ \left[ \{J_{m-1}(0, z_1) + J_{m+1}(0, z_1)\} J_n(0, z_2) J_p(0, z_3) + \delta_1 \{J_{n-1}(0, z_2) + J_{n+1}(0, z_2)\} J_m(0, z_1) J_p(0, z_3) \right] \\ + \delta_2 [J_{p-1}(0, z_3) + J_{p+1}(0, z_3)] J_m(0, z_1) J_n(0, z_2) \quad (15)$$

The gain is

$$G = \frac{j}{v_0^3} [2 - 2 \cos(v_0) - v_0 \sin(v_0)] \quad (16)$$

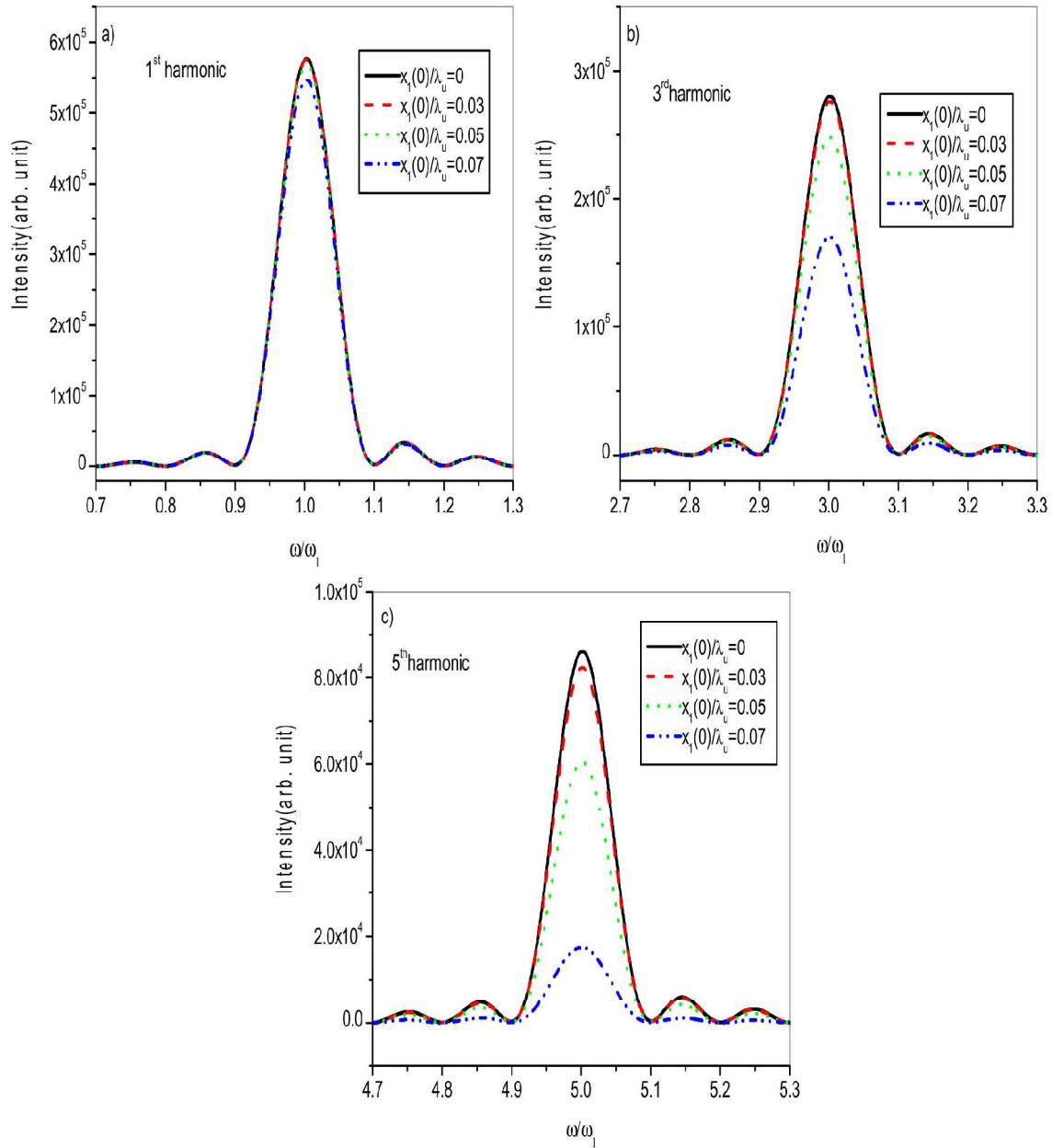
(1)

#### 4. Results & Discussion

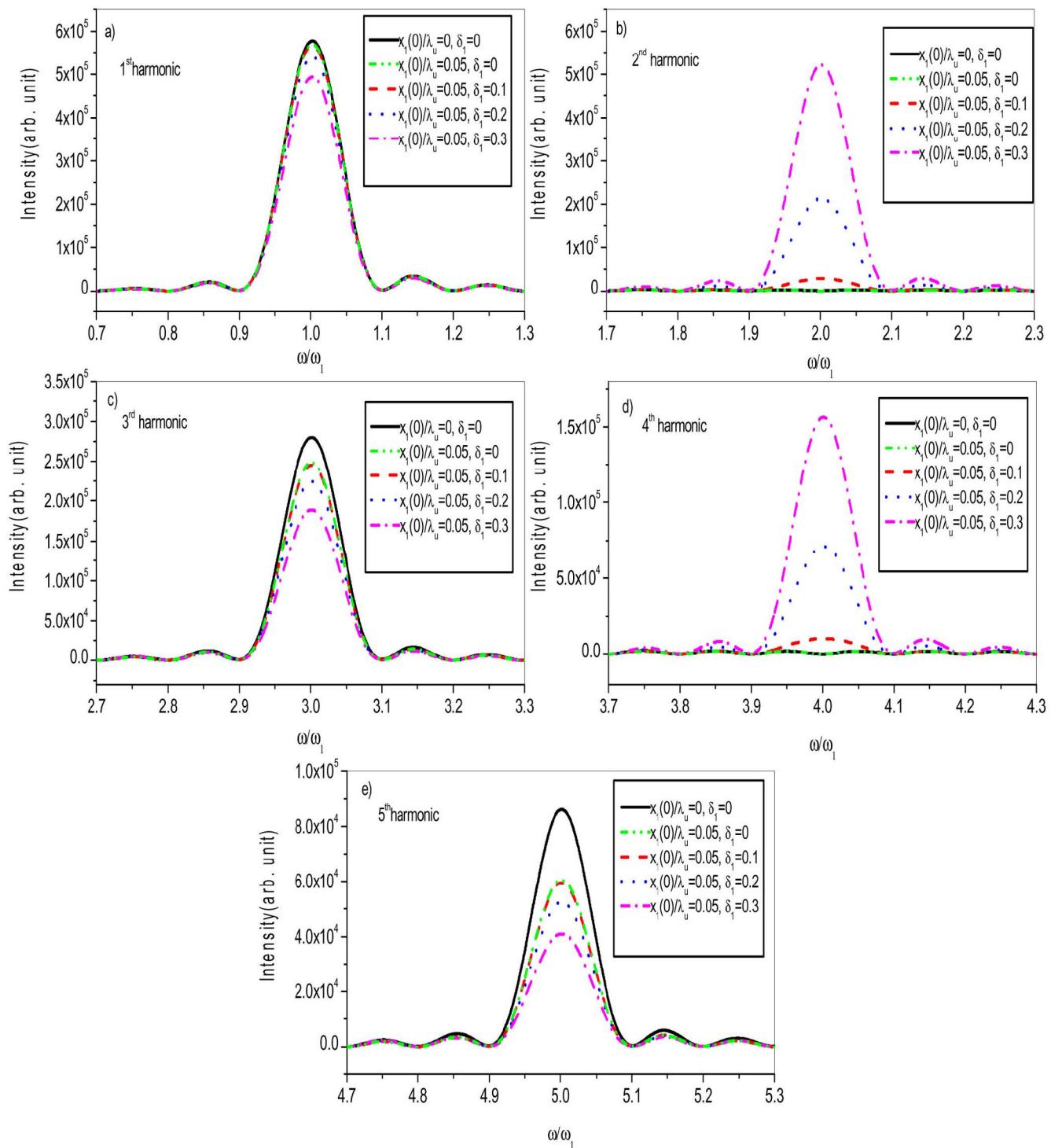
In this paper we have examined the use of three frequency undulator scheme for reducing betatron oscillation effects in a free electron laser. In **Figure.(1)** the linear undulator emits at odd harmonics ( $m=1, 3, 5, \dots$ ). The betatron oscillation reduces the intensity at respective harmonics. **Figure (2)** represents both odd and even harmonics due to inclusion of  $2\omega_u$  frequency. Here we saw the intensity enhancement for even harmonics as we increase value of  $\delta_1$  and odd harmonic intensity decreases. In **Figure(3)** we saw the intensity of third and fifth harmonic increases and we reach intensity of third harmonic as comparable to first harmonic at  $\delta_2 = 0.3$ . **Figure (4)** represents intensity plots for various harmonics and here we find that intensity of all the higher harmonics either odd or even get enhanced with certain combination of  $\delta_1$  and  $\delta_2$ . In **Figure (5)** we plot gain curves versus frequency. **Figure (5a)** shows gain degradation due to inclusion of betatron motion of electron.

**Figure (5b)** enhances gain for even harmonics and **Figure (5c)** enhances gain for odd harmonics. **Figure (5d)** enhances gain for odd and even harmonics except first harmonic for various values of  $\delta_1$  and  $\delta_2$ . The role of three frequency undulator scheme concludes that we can achieve higher intensity and gain at higher harmonics.

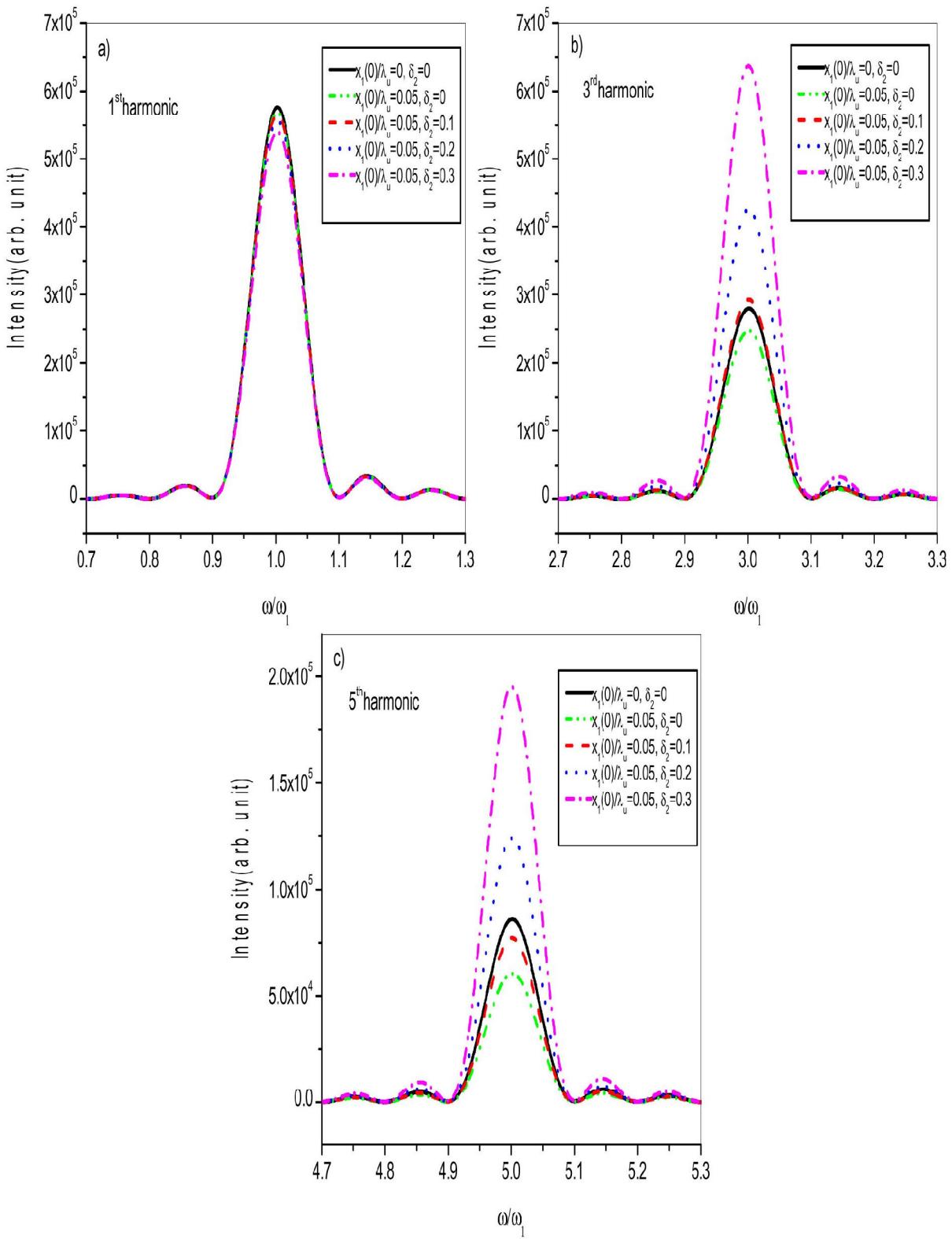
The analysis reflects the view that the betatron oscillations reduce the intensity and gain. However, in the case of a three frequency undulator, one gets substantial enhanced intensity and gain at a selected harmonic in comparison to a regular planar undulator field.

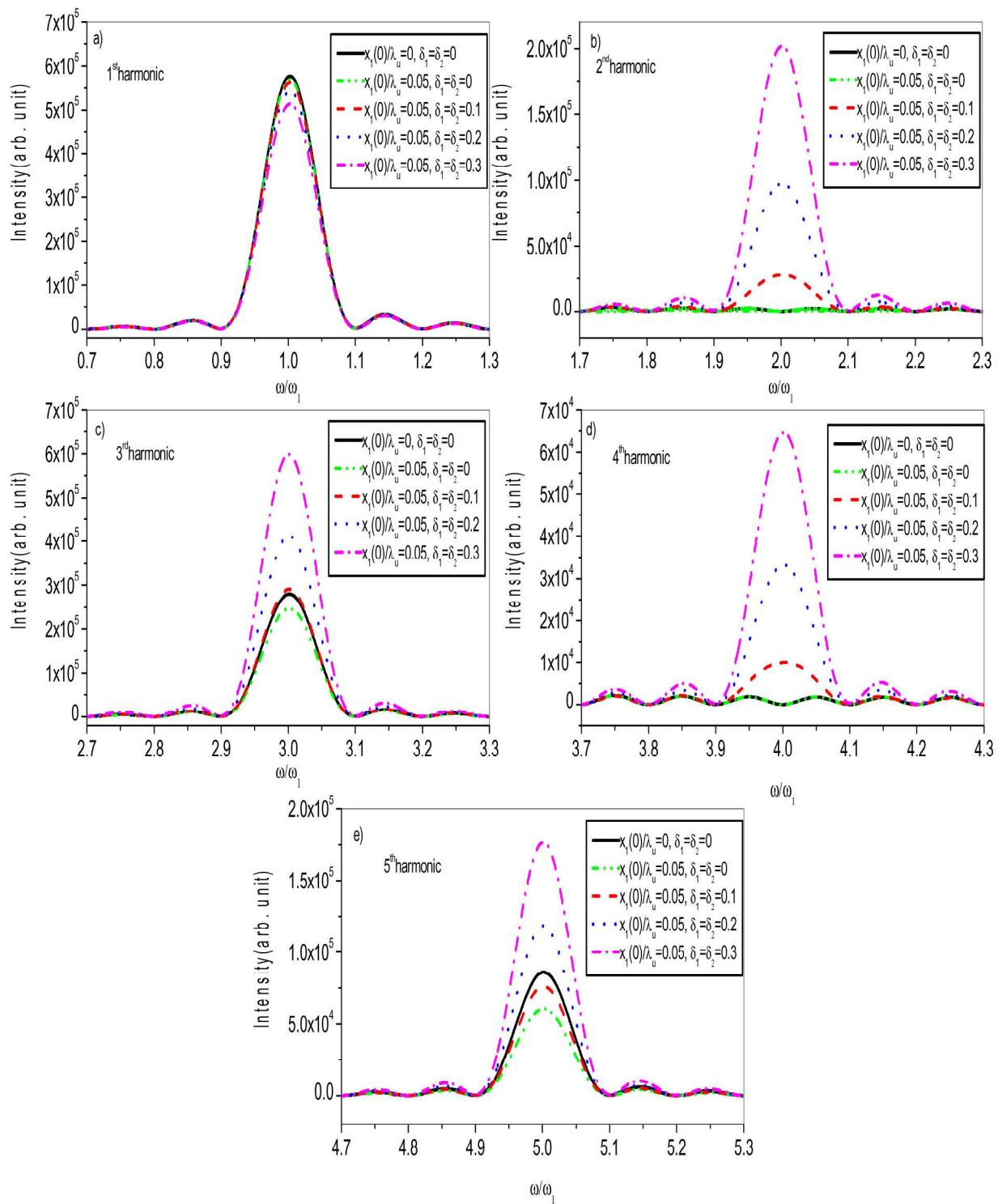


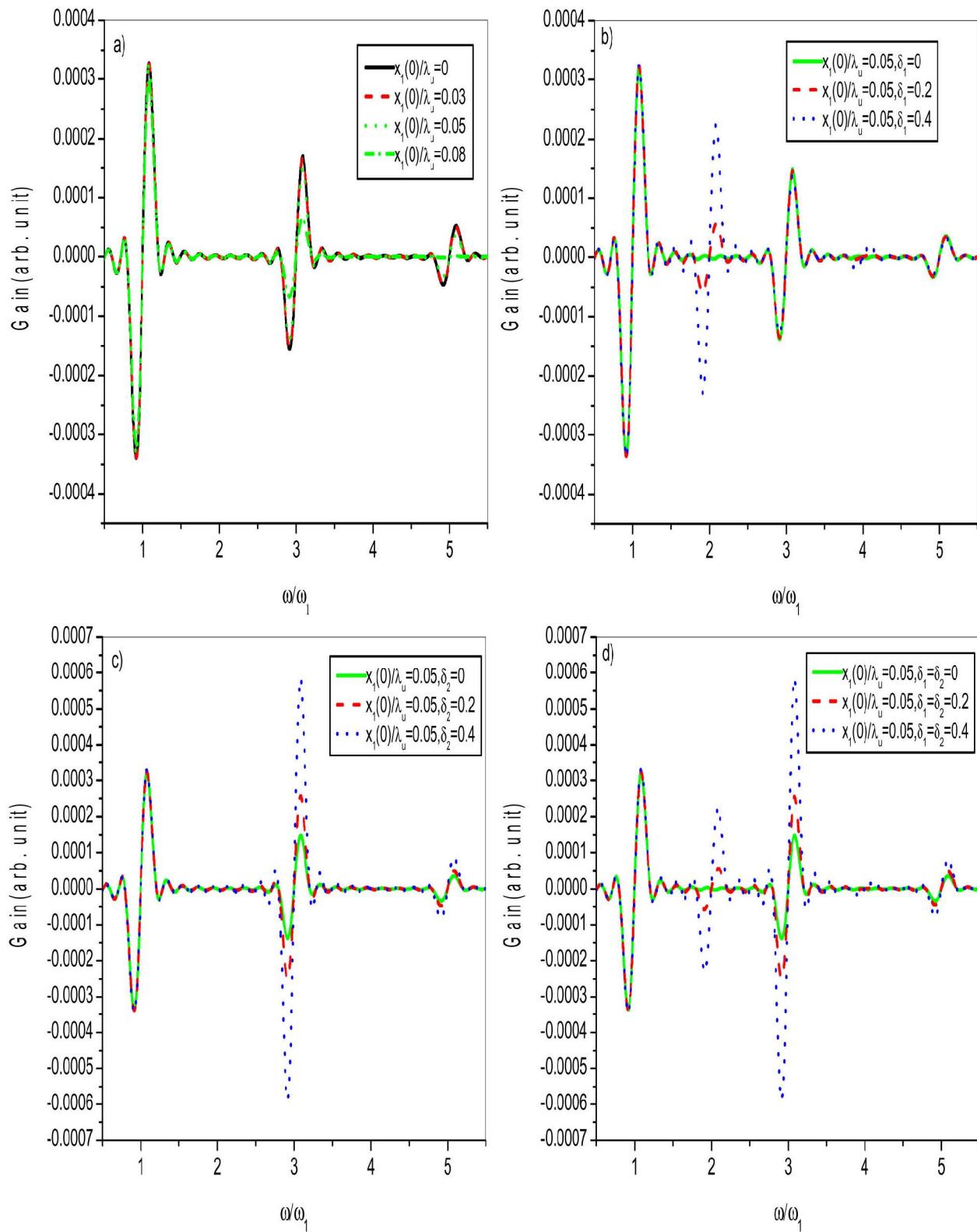
**Figure 1.** Intensity *vs.* frequency with various values of  $\frac{x_1(0)}{\lambda_u}$ .



**Figure 2.** Intensity *vs.* frequency for various values of  $\delta_1$  .

**Figure 3 .** Intensity *vs.* frequency for various values of  $\delta_2$

**Figure 4.** Intensity *vs.* frequency for various values of  $\delta_1$  and  $\delta_2$  .

**Figure 5.** Gain curve versus frequency for various harmonics.

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